Direct and inverse scattering problems for quasi-linear biharmonic operator in 3D

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1 Introduction

In scattering theory, a plane wave is sent from a known direction to some object. We call this object a scatterer. Interaction between the incident wave and the scatterer forms a new wave which is called a scattered wave. Direct scattering problem concerns determining this scattered wave when the scatterer is known. Conversely in inverse scattering problem the goal is to gather information about the scatterer while the behaviour of the scattered wave is known. Such information can be for example the shape of the object. Mathematically the scattering problem can be formulated as a partial differential equation. The partial differential operator determines the types of these waves and the scatterer is presented as a perturbation to the wave. The most commonly studied operator is Schrödinger operator and some recent studies of that can be found in [1, 2, 5, 7].
We consider a three-dimensional biharmonic operator

$$H_4 = \Delta^2 + V,$$

where Δ is Laplacian and potential V is a scalar valued quasi-linear function. Conditions for function V will be specified later. The scattering problem for this operator is given by

$$H_4 u = k^4 u, \quad u = u_0 + u_{sc},$$

where the coefficient k corresponds to the wave number. The solution u is assumed to be a sum of two function u_0 and u_{sc}, where the function u_0 is a plane wave and u_{sc} is an outgoing wave in the sense that it satisfies to the Sommerfeld radiation conditions for biharmonic operator at the infinity. While Schrödinger equation can be used to model the behaviour of waves, strings and particles, biharmonic operators appear in the studies of elasticity and vibrations of beams. Scattering theory of biharmonic operators with linear perturbations is studied before in [9, 10, 11] and with quasi-linear perturbations on the line in [12]. We will follow [11] and the main results will correspond to those obtained in [11]. This text focuses on the direct scattering problem, but also two results regarding the inverse problem will be given. Main results of this thesis are asymptotic behaviour of the solution and the proof of Saito's formula. We will give two results regarding the inverse problem, namely uniqueness and representation formula for the unknown potential V. They both follow from Saito's formula and thus Saito's formula can be thought as a bridge between the two problems.

This text is organized as follows. We start by fixing some notations and introducing definitions and known results from various areas of mathematics. Those results will be used later in the text. After that we formulate the scattering problem as a partial differential equation which we then turn into a Lippmann-Schwinger integral equation. It will be shown that with some assumptions the solution to the differential equations indeed is a solution to the Lippmann-Schwinger equation. The solvability of Lippmann-Schwinger equation will be proved. We then proceed to define the scattering amplitude and show that the asymptotic behaviour of the solution can be expressed using the scattering amplitude. Finally, we will prove Saito's formula and two of its corollaries. We conclude the text by introducing the inverse backscattering Born approximation and discuss some results that have been obtained using it.

2 Preliminaries

2.1 Function spaces

We start by defining some function spaces that we will use throughout the text. We will use same definitions and notations as in [6]. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. We define the Lebesgue spaces $L^p(\Omega)$ for $1 \leq p < \infty$ by

$$L^p(\Omega) := \{f : \Omega \to \mathbb{C} \text{ is measurable} : \|f\|_p = \Big(\int_\Omega |f(x)|^p dx\Big)^{1/p} < \infty\}$$

and

$$L^\infty(\Omega) := \{f : \Omega \to \mathbb{C} \text{ is measurable} : \|f\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| < \infty\}.$$

Let us formulate two theorems regarding Lebesgue spaces.

Theorem 2.1. *If $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ where $1 \leq p, p' \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then the product fg is integrable and*

$$\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_{p'}.$$

Theorem 2.2. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} e^{-i(x,y)} f(y) dy \to 0, \text{ as } |x| \to \infty.$$

Theorem 2.1 is called Hölder inequality and Theorem 2.2 is known as Riemann-Lebesgue lemma. In particular Hölder inequality will be used frequently in the future. Proofs for these two theorems can be found in [3, p. 96] and [6, p. 399-400], respectively.

Let us introduce the following notations. We say that α is a multi-index if $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_i \in \mathbb{N}_0$, for all $i = 1, 2, 3$. We denote this by $\alpha \in \mathbb{N}_0^3$. For higher order derivatives we use the shorthand notation ∂^α which is given by

$$\partial^\alpha = \frac{\partial^{\alpha 1+\alpha_2+\alpha_3}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

For $k \in \mathbb{N}_0$ and $1 \leq p < \infty$ we define the Sobolev space $W_p^k(\mathbb{R}^3)$ as

$$W_p^k(\mathbb{R}^3) = \Big\{ f \in L^p(\mathbb{R}^3) : \|f\|_{W_p^k(\mathbb{R}^3)} := \Big(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_p^p \Big)^{1/p} < \infty \Big\}.$$

The space of $n \in \mathbb{N}$ times continuously differentiable functions is given by

$$C^n(\mathbb{R}^3) = \{ f : \mathbb{R}^3 \to \mathbb{C} : \partial^\alpha f(x) \text{ exists and is continuous for all } |\alpha| \leq n \}.$$

We say that f is a smooth function or $f \in C^\infty(\mathbb{R}^3)$ if $f \in C^n(\mathbb{R}^3)$ for all $n \in \mathbb{N}$.

The space of compactly supported smooth functions is given by

$$C_0^\infty(\mathbb{R}^3) = \Big\{ f : \mathbb{R}^n \to \mathbb{C} \ : \operatorname{supp} f \text{ is compact and} f \in C^\infty(\mathbb{R}^3) \Big\},$$

where $\operatorname{supp} f = \overline{\{x \in \mathbb{R}^3 : \ f(x) \neq 0\}}$ is called the support of function f. The Schwartz space of rapidly decaying functions is defined as

$$S(\mathbb{R}^3) = \Big\{ f \in C^\infty(\mathbb{R}^3) : |f|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^3} \Big| x^\alpha \partial^\beta f(x) \Big| < \infty, \ \text{ for any } \alpha, \beta \in \mathbb{N}_0^3 \Big\}.$$

Without going into details about vector fields we give the following two results.

Theorem 2.3 (Divergence theorem)**.** *Let the boundary of domain $\Omega \subset \mathbb{R}^3$ be C^1 and let us denote it by $\partial\Omega$. If F is a C^1-vector field on the closure $\overline{\Omega}$, then*

$$\int_\Omega \nabla \cdot F dx = \int_{\partial\Omega} F \cdot \nu d\sigma(x).$$

Here ν is an outward normal vector on the surface $\partial\Omega$.

The proof of Theorem 2.3 for convex region $\Omega \subset \mathbb{R}^3$ with no holes as well as some justification for the general case can be found in [8, p.974-976].

Theorem 2.4 (Green's second identity)**.** *Let the boundary of the domain $\Omega \subset \mathbb{R}^3$ be as in Theorem 2.3. If $f, g \in C^2(\Omega) \cap C^1(\overline{\Omega})$, then*

$$\int_\Omega (g \Delta f + \nabla g \nabla f) dx = \int_{\partial\Omega} g \partial_\nu f d\sigma(x),$$

where ∂_ν is the normal derivative with respect to the surface.

Proof. Let $F = g\nabla f$. Using the divergence theorem and Leibniz rule for differentiation we have that

$$\int_{\Omega} (g\Delta f + \nabla g \nabla f)dx = \int_{\Omega} \nabla \cdot F dx = \int_{\partial\Omega} g\partial_\nu f d\sigma(x).$$

$\square$

Note that is $\Omega = B(0,R) = \{x \in \mathbb{R}^3 : |x| < R\}$ for some $R > 0$, then $\partial_\nu = \frac{\partial}{\partial |x|}$ on $\partial\Omega$.

2.2 Fourier transform

Definition 2.5. Let $f \in S(\mathbb{R}^3)$. We define the Fourier transform of function f as

$$\mathcal{F}f(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i(x,\xi)} f(x)dx, \quad \xi \in \mathbb{R}^3.$$

Inverse Fourier transform $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}f(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i(\xi,x)} f(\xi)d\xi.$$

For one-dimensional inverse Fourier transform it is proved in [6, p.85] that

$$\mathcal{F}^{-1}\mathcal{F}f(x) = \mathcal{F}\mathcal{F}^{-1}f(x) = f(x)$$

and the same is true for three-dimensional inverse Fourier transform.

Definition 2.6. The convolution of two functions $f, g \in S(\mathbb{R}^3)$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}^3} f(x-y)g(y)dy.$$

Note that the convolution is symmetric, i.e., $f * g = g * f$.

Theorem 2.7. *Let $f, g \in S(\mathbb{R}^3)$. Then*

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{3/2}\mathcal{F}(f)(\xi)\mathcal{F}(g)(\xi)$$

and conversely

$$\mathcal{F}^{-1}(f \cdot g)(x) = (2\pi)^{-3/2}(\mathcal{F}^{-1}(f) * \mathcal{F}^{-1}(g))(x).$$

Proof. This theorem was proved in [6, p.170] $\square$

2.3 Distributions and fundamental solution

Later in this text we will come across some functions that either behave badly or we do not know enough about their behaviour. Rather than considering them as regular functions, we can study how they act on some group of test function as functionals. This type of study is called distribution theory. We will define two types of distributions, Schwartz distribution and tempered distributions.

Let us start by defining the null-sequence. This leads us to the concept of continuity for functionals.

Definition 2.8. A sequence $\{\varphi_j\}_{j=0}^{\infty}$ of $C_0^\infty(\mathbb{R}^3)$-functions is called a null-sequence if

1. there exists a compact set $K \subset \mathbb{R}^3$ such that $\operatorname{supp}\varphi_j \subset K$, for all $j \in \mathbb{N}_0$
2. for every $\alpha \geq 0$ we have

$$\sup_{x\in K} |\partial^\alpha \varphi_j(x)| \to 0,\ j \to \infty.$$

We use the notation $\langle T, \varphi\rangle$ to denote the application of the functional T to the function φ, i.e., $T(\varphi) = \langle T, \varphi\rangle$.

Definition 2.9. A functional $T : C_0^\infty(\mathbb{R}^3) \to \mathbb{C}$ is said to be a Schwartz distribution if it is linear and continuous, i.e.

1. $\langle T, \alpha\varphi+\beta\psi\rangle = \alpha\langle T, \varphi\rangle+\beta\langle T, \psi\rangle$, for every $\varphi, \psi \in C_0^\infty(\mathbb{R}^3)$ and $\alpha, \beta \in \mathbb{C}$
2. for every null-sequence $\varphi_j \in C_0^\infty(\mathbb{R}^3)$ we have $\langle T, \varphi_j\rangle \to 0$ in $\mathbb{C}$, as $j \to \infty$.

The linear space of Schwartz distribution is denoted by $\mathscr{D}'$.

Definition 2.10. A functional $T : S(\mathbb{R}^3) \to \mathbb{C}$ is said to be a tempered distribution if

1. T is linear, $\langle T, \alpha\varphi + \beta\psi\rangle = \alpha\langle T, \varphi\rangle + \beta\langle T, \psi\rangle$
2. T is continuous on $S(\mathbb{R}^3)$, in other words there exists $n_0 \in \mathbb{N}_0$ and a constant $c_0 > 0$ such that

$$|\langle T, \varphi\rangle| \leq c_0 \sum_{|\alpha|,|\beta|\leq n_0} |\varphi|_{\alpha,\beta}.$$

The space of tempered distributions is denoted by S'.

Note that because of the embedding $C_0^\infty(\mathbb{R}^3) \subset S(\mathbb{R}^3)$, the space of Schwartz distributions is wider than the space of tempered distributions. From these definitions it follows that every locally integrable function $f \in L^1_{loc}(\mathbb{R}^3)$ defines a Schwartz distribution via the formula

$$\langle f, \varphi \rangle = \int_{\operatorname{supp} \varphi} f(x)\varphi(x)dx.$$

Since the Fourier transform was defined as an operator from $S(\mathbb{R}^3)$ to itself, we can define the Fourier transform of a tempered distribution T as a functional satisfying

$$\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle, \quad \text{for all } \varphi \in S(\mathbb{R}^3).$$

Similarly we define the derivative of T by

$$\langle \partial^\alpha T, \varphi \rangle = \langle T, (-1)^{|\alpha|}\partial^\alpha \varphi \rangle, \quad \text{for all } \varphi \in S(\mathbb{R}^3).$$

Before we define the fundamental solution of partial differential operator, we introduce the Dirac delta distribution δ_{x_0}. It is a distribution satisfying

$$\langle \delta_{x_0}, \psi \rangle = \psi(x_0), \quad \text{for all}\, \psi \in C_0^\infty(\mathbb{R}^3).$$

We have the basic properties

$$\begin{aligned}
&\mathcal{F}(\delta_0) = (2\pi)^{-3/2} \cdot 1 \\
&\mathcal{F}(1) = (2\pi)^{3/2}\delta_0 \\
&\mathcal{F}(x^\alpha) = i^{|\alpha|}\partial^\alpha(\mathcal{F}(1)) = i^{|\alpha|}(2\pi)^{3/2}\partial^\alpha\delta_0.
\end{aligned}$$

Definition 2.11. Let L be a partial differential operator. A fundamental solution of L is a Schwartz distribution K such that for all $\varphi \in C_0^\infty(\mathbb{R}^3)$ we have,

$$\langle LK, \varphi \rangle = \varphi(0), \quad \text{or in other words} \quad LK = \delta_0.$$

Let T be a tempered distribution and $\Omega \subset \mathbb{R}^3$ be an open set. We say that T vanishes on Ω if $\langle T, \psi \rangle = 0$, for all functions $\psi \in S(\mathbb{R}^3)$ with $\operatorname{supp} \psi \subset \Omega$.

Definition 2.12. Let $\Omega \subset \mathbb{R}^3$ and T is a distribution on Ω. We define the support of T as the complement of the largest open set on which T vanishes.

Theorem 2.13. *Let T be a distribution with single point support,* $\operatorname{supp} T = \{x_0\}$. *Then there exists an integer $N \in \mathbb{N}$ and complex numbers C_α such that*

$$T = \sum_{|\alpha| \le N} C_\alpha \partial^\alpha \delta_{x_0}.$$

Proof. See [4, Prop. 2.4.1, p.124-125] □

2.4 Useful results from functional analysis

Here is a collection of some results from functional analysis and operator theory that we will use in the future.

Definition 2.14. Let X be a normed space. We say that a mapping $A : X \to X$ is a contraction mapping, if there exists $0 < \tau < 1$ such that

$$||A(x) - A(y)||_X \leq \tau ||x - y||_X, \quad \textit{for all } x, y \in X.$$

Theorem 2.15. *Let X be a complete metric space and $\Omega \subset X$ a closed subset. If function $f : \Omega \to \Omega$ is a contraction mapping, then there exists unique $\tilde{x} \in \Omega$ such that $f(\tilde{x}) = \tilde{x}$. Moreover this fixed point can be found as*

$$\lim_{j\to\infty} x_j = \tilde{x},$$

where $x_0 \in \Omega$ is an arbitrary point and $x_j = f(x_{j-1})$.

This theorem is called Banach fixed-point theorem and its proof can be found in [3, p. 119-120].

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then

$$Af(x) = \int_\Omega K(x, y) f(y) dy$$

is an integral operator in $L^2(\Omega)$ with kernel K.

Definition 2.16. Integral operator A is said to be an operator with weak singularity if its kernel $K(x, y)$ is continuous for all $x, y \in \mathbb{R}^n, x \neq y$ and there are positive constants M and $\alpha \in (0, n]$ such that

$$|K(x, y)| \leq M|x - y|^{-\alpha}.$$

For two integral operators

$$A_1 f(x) = \int_\Omega K_1(x, y) f(y) dy \quad \text{and } A_2 f(x) = \int_\Omega K_2(x, z) f(z) dz$$

we define

$$(A_1 \circ A_2) f(x) = \int_\Omega K_1(x, y) A_2 f(y) dy = \int_\Omega K_1(x, y) \Big(\int_\Omega K_2(y, z) f(z) dz \Big) dy$$
$$\int_\Omega \Big(\int_\Omega K_1(x, y) K_2(y, z) dy \Big) f(z) dz = \int_\Omega K(x, z) f(z) dz$$

and

$$(A_2 \circ A_1) f(x) = \int_\Omega K_2(x, z) A_1 f(z) dz = \int_\Omega K_2(x, z) \Big(\int_\Omega K_1(z, y) f(y) dy \Big) dz$$
$$= \int_\Omega \Big(\int_\Omega K_2(x, z) K_1(z, y) dz \Big) f(y) dy = \int_\Omega \widetilde{K}(x, y) f(y) dy.$$

Theorem 2.17. *If A_1 and A_2 are integral operators with weak singularities then $A_1 \circ A_2$ as well as $A_2 \circ A_1$ are also integral operators with weak singularities. More precisely, if*

$$|K_1(x,y)| \leq M_1|x-y|^{-\alpha_1} \quad \text{and} \quad |K_2(x,y)| \leq M_2|x-y|^{-\alpha_2},$$

where $\alpha_1, \alpha_2 \in (0,n]$, then there exists $M > 0$ such that

$$|K(x,y)| \leq M \begin{cases} |x-y|^{n-\alpha_1-\alpha_2}, & \alpha_1+\alpha_2 > n \\ 1+|\log|x-y||, & \alpha_1+\alpha_2 = n \\ 1, & \alpha_1+\alpha_2 < n, \end{cases}$$

where $K(x,y)$ is the kernel of operator $A_1 \circ A_2$. Same estimate holds for the kernel $\widetilde{K}(x,y)$ of operator $A_2 \circ A_1$.

Proof. The proof can be found in [6, p.360-362] □

Remark 2.18. In the case where $\alpha_1 + \alpha_2 > n$, Theorem 2.17 holds also when Ω is not bounded, in particular it holds when $\Omega = \mathbb{R}^n$.

3 Direct scattering problem

Let us introduce the three-dimensional biharmonic operator H_4 by setting

$$H_4 u(x) = \Delta^2 u(x) + V(x,|u|)u(x), \tag{1}$$

where Δ is the Laplacian and V is a scalar-valued function depending on both $x \in \mathbb{R}^3$ and on the modulus of function u, meaning that biharmonic operator is perturbed by a quasi-linear perturbation of order zero.
The scattering problem for this operator is given by

$$\begin{cases} H_4 u(x,k,\theta) = k^4 u(x,k,\theta), \quad x \in \mathbb{R}^3, \ k>0, & (2) \\ u(x,k,\theta) = u_0(x,k,\theta) + u_{sc}(x,k,\theta), & (3) \\ \lim\limits_{r\to\infty} r\left[\dfrac{\partial f}{\partial r} - ikf\right] = 0, \text{ for both } f = u_{sc} \text{ and } f = \Delta u_{sc}. & (4) \end{cases}$$

Here the coefficient $k > 0$ corresponds to the wave number and $u_0(x,k,\theta) = e^{ik(x,\theta)}$ is the incident wave coming from the direction $\theta \in \mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$. The regular real inner product of vectors $x, y \in \mathbb{R}^3$ is denoted by (x,y). We are interested in solutions u_{sc} with finite modulus, i.e., $u_{sc} \in L^\infty(\mathbb{R}^3)$.
By rearranging the equation (2), we obtain

$$\Delta^2 u(x,k,\theta) - k^4 u(x,k,\theta) = -V(x,|u|)u(x,k,\theta). \tag{5}$$

The fundamental solution of operator $H_0 = \Delta^2 - k^4$ in n-dimensions is known to be

$$G_k^+(|x|) = \frac{i}{8k^2}\Big(\frac{|k|}{2\pi|x|}\Big)^{\frac{n-2}{2}}\Big(H^{(1)}_{\frac{n-2}{2}}(|k||x|) + \frac{2i}{\pi}K_{\frac{n-2}{2}}(|k||x|)\Big),$$

where $H^{(1)}_{\frac{n-2}{2}}$ and $K_{\frac{n-2}{2}}$ are Hankel function of the first kind and Macdonald's function of orders $\frac{n-2}{2}$. Some justification for this can be found in [11]. In three-dimensional case orders of these functions are $\frac{1}{2}$ and cylinder functions of half-integer order can be expressed as elementary functions. This simplifies to

$$G_k^+(|x|) = \frac{e^{ik|x|} - e^{-k|x|}}{8\pi k^2|x|}, \quad \text{in } \mathbb{R}^3.$$

By applying the fundamental solution via convolution to the equation (5), we obtain a Lippmann-Schwinger integral equation

$$u(x,k,\theta) = u_0(x,k,\theta) - \int_{\mathbb{R}^3} G_k^+(|x-y|)V(y,|u|)u(y,k,\theta)dy. \tag{6}$$

Lemma 3.1. *For any $x \in \mathbb{R}^3$ and k large enough, we have that*

$$|G_k^+(|x|)| \leq \frac{1}{2\pi k}.$$

Proof. We will consider three cases. First let us assume that $|x| > 1$. Now

$$|G_k^+(|x|)| = \frac{|e^{ik|x|} - e^{-k|x|}|}{8\pi k^2|x|} \leq \frac{2}{8\pi k^2}.$$

When $0 < |x| < 1$ and $k|x| > 1$, we have inequality $1/|x| < k$ and therefore

$$|G_k^+(|x|)| \leq \frac{2}{8\pi k}.$$

Finally when $0 < k|x| < 1$, by using the Taylor expansion of $e^{ik|x|} - e^{-k|x|}$ we have that

$$\begin{aligned}|e^{ik|x|} - e^{-k|x|}| &\leq \sum_{j=1}^{\infty} \frac{|i^j - (-1)^j|}{j!}(k|x|)^j \leq 2k|x|\Big(\sum_{j=0}^{\infty}\frac{1}{j!} - 1\Big)\\ &= 2(e-1)k|x| \leq 4k|x|.\end{aligned}$$

From this it follows that

$$|G_k^+(|x|)| \leq \frac{4k|x|}{8\pi k^2|x|} = \frac{1}{2\pi k}.$$

For $k > 0$ large enough this last estimate gives us the largest upper bound for $|G_k^+(|x|)|$ and therefore Lemma 3.1 is proved. □

3.1 From differential equation to integral equation

In this section we will show that a solution to (2) indeed satisfies the equation (6). The same study was done in [11] for biharmonic operator with linear perturbations and those results can be used in our problem. We proceed as in [11].

Lemma 3.2. *Let us assume that the function $V(\cdot, s) \in L^1(\mathbb{R}^3)$ with respect to the first argument $x \in \mathbb{R}^3$ for all $s < \infty$. If the function $u = u_0 + u_{sc}$, $u_{sc} \in W^4_\infty(\mathbb{R}^3)$ is a solution to (2) with fixed $k > 0$, then there exists a constant $C > 0$ such that*

$$\lim_{R\to\infty} \int_{|y|=R} (|u_{sc}|^2 + |\Delta u_{sc}|^2) d\sigma(y) \leq C.$$

Proof. By first using Sommerfeld radiation conditions and then Green's second identity, we have

$$\begin{aligned}
&2ik \int_{|y|=R} (|\Delta u_{sc}|^2 + k^4 |u_{sc}|^2) d\sigma(y) \\
&= \int_{|y|=R} (\overline{\Delta u_{sc}}(ik\Delta u_{sc}) - \Delta u_{sc}\overline{(ik\Delta u_{sc})} + k^4\overline{u_{sc}}(iku_{sc}) - k^4 u_{sc}\overline{(iku_{sc})}) d\sigma(y) \\
&= \int_{|y|=R} \Big(\overline{\Delta u_{sc}}\frac{\partial}{\partial n}\Delta u_{sc} - \Delta u_{sc}\frac{\partial}{\partial n}\overline{\Delta u_{sc}} + k^4\overline{u_{sc}}\frac{\partial}{\partial n}u_{sc} - k^4 u_{sc}\frac{\partial}{\partial n}\overline{u_{sc}}\Big) d\sigma(y) \\
&+ o(R^{-1}) \int_{|y|=R} (\Delta u_{sc} + \Delta\overline{u_{sc}} + k^4 u_{sc} + k^4\overline{u_{sc}}) d\sigma(y) \\
&= \int_{|y|\leq R} (\overline{\Delta u_{sc}}\Delta^2 u_{sc} - \Delta u_{sc}\overline{\Delta^2 u_{sc}} + k^4\overline{u_{sc}}\Delta u_{sc} - k^4 u_{sc}\overline{\Delta u_{sc}}) dy \\
&+ o(R^{-1}) \int_{|y|=R} (\Delta u_{sc} + \Delta\overline{u_{sc}} + k^4 u_{sc} + k^4\overline{u_{sc}}) d\sigma(y) \\
&= \int_{|y|\leq R} (\overline{\Delta u_{sc}}(-Vu) - \Delta u_{sc}(-Vu)) dy \\
&+ o(R^{-1}) \int_{|y|=R} (\Delta u_{sc} + \Delta\overline{u_{sc}} + k^4 u_{sc} + k^4\overline{u_{sc}}) d\sigma(y),
\end{aligned}$$

where we have used the fact that $\Delta^2 u_{sc} - k^4 u_{sc} = -Vu$.
The first integral above is finite due to $\Delta u_{sc}, u \in L^\infty(\mathbb{R}^3)$ and $V \in L^1(\mathbb{R}^3)$. The second integral we consider in two parts and estimate them by modulus.

First, by Hölder inequality we have

$$\begin{aligned}
o(R^{-1})\int_{|y|=R}|\Delta u_{sc}+\Delta\overline{u_{sc}}|d\sigma(y) &= o(R^{-1})\int_{|y|=R}|\operatorname{Re}(\Delta u_{sc})|d\sigma(y)\\
&\le o(R^{-1})\Big(\int_{|y|=R}1d\sigma(y)\Big)^{1/2}\Big(\int_{|y|=R}|\operatorname{Re}(\Delta u_{sc})|^2d\sigma(y)\Big)^{1/2}\\
&\le o(1)\Big(\int_{|y|=R}|\Delta u_{sc}|^2d\sigma(y)\Big)^{1/2}.
\end{aligned}$$

Similarly

$$o(R^{-1})\int_{|y|=R}|k^4u_{sc}+k^4\overline{u_{sc}}|d\sigma(y)=o(1)\Big(\int_{|y|=R}|u_{sc}|^2d\sigma(y)\Big)^{1/2}.$$

Combining these facts gives us that

$$\begin{aligned}
\int_{|y|=R}(|\Delta u_{sc}|^2+|u_{sc}|^2)d\sigma(y) \le& C\|\Delta u_{sc}\|_\infty\|u\|_\infty\|V\|_1\\
&+o(1)\Big(\int_{|y|=R}(|\Delta u_{sc}|^2+|u_{sc}|^2)d\sigma(y)\Big)^{1/2}.
\end{aligned}$$

This estimate proofs the lemma. □

Lemma 3.3. *Let $\Omega\subset\mathbb{R}^3$ be an open and bounded domain with smooth boundary $\partial\Omega$. If $f,g\in W_2^4(\Omega)$ then the following equality holds*

$$\int_\Omega(f\Delta^2g-g\Delta^2f)dx=\int_{\partial\Omega}\Big(f\frac{\partial}{\partial n}\Delta g+\Delta f\frac{\partial}{\partial n}g-g\frac{\partial}{\partial n}\Delta f-\Delta g\frac{\partial}{\partial n}f\Big)d\sigma(x).$$

Proof. This lemma follows from the divergence theorem when $f,g\in C^4(\Omega)$ and we define the C^1-vector field F as

$$F=\Delta f\nabla g+f\nabla(\Delta g)-\Delta g\nabla f-g\nabla(\Delta f).$$

□

Theorem 3.4. *If the function u_{sc} satisfies to the conditions of Lemma 3.2 then it is a solution to the equation* (6).

Proof. Let $x\in\mathbb{R}^3$ be a fixed point and $R>0$ so that $x\in B(0,R)$. Let then $\varepsilon>0$ such that $B(x,\varepsilon)\subset B(0,R)$ and denote $\Omega_{R,\varepsilon}=B(0,R)\backslash B(x,\varepsilon)$.

Using Lemma 3.3 we can calculate

$$\begin{aligned}
&\int_{\Omega_{R,\varepsilon}} \Big(u_{sc}(y)(\Delta_y^2 - k^4)G_k^+(|x-y|) - G_k^+(|x-y|)(\Delta^2 - k^4)u_{sc}(y)\Big)dy \\
&= \int_{\Omega_{R,\varepsilon}} \Big(u_{sc}\Delta_y^2 G_k^+(|x-y|) - G_k^+(|x-y|)\Delta^2 u_{sc}(y)\Big)dy \\
&= \int_{\partial\Omega_{R,\epsilon}} \Big[u_{sc}\frac{\partial}{\partial n}\Delta_y G_k^+(|x-y|) + \Delta u_{sc}\frac{\partial}{\partial n}G_k^+(|x-y|) \\
&\quad - G_k^+(|x-y|)\frac{\partial}{\partial n}\Delta u_{sc} - \Delta_y G_k^+(|x-y|)\frac{\partial}{\partial n}u_{sc}\Big]d\sigma(y) \\
&= \int_{\partial\Omega_{R,\varepsilon}} \Big[u_{sc}\Big(\frac{\partial}{\partial n} - ik\Big)\Delta_y G_k^+(|x-y|) + \Delta u_{sc}\Big(\frac{\partial}{\partial n} - ik\Big)G_k^+(|x-y|) \\
&\quad - G_k^+(|x-y|)\Big(\frac{\partial}{\partial n} - ik\Big)\Delta u_{sc} - \Delta_y G_k^+(|x-y|)\Big(\frac{\partial}{\partial n} - ik\Big)u_{sc}\Big]d\sigma(y).
\end{aligned}$$

Since G_k^+ is the fundamental solution of $\Delta^2 - k^4$, by letting $\varepsilon \to 0$ we obtain

$$\begin{aligned}
u_{sc}(x) = &- \int_{\Omega_R} G_k^+(|x-y|)V(y,|u|)u(y)dy \\
&+ \int_{\partial\Omega_R} \Big[u_{sc}\Big(\frac{\partial}{\partial n} - ik\Big)\Delta_y G_k^+(|x-y|) + \Delta u_{sc}\Big(\frac{\partial}{\partial n} - ik\Big)G_k^+(|x-y|) \\
&- G_k^+(|x-y|)\Big(\frac{\partial}{\partial n} - ik\Big)\Delta u_{sc} - \Delta_y G_k^+(|x-y|)\Big(\frac{\partial}{\partial n} - ik\Big)u_{sc}\Big]d\sigma(y).
\end{aligned} \tag{7}$$

Now because

$$\lim_{R\to\infty} \int_{\partial\Omega_R} |f(y)|^2 d\sigma(y) \le C, \quad \text{for both } f = u_{sc} \text{ and } f = \Delta u_{sc},$$

by Hölder inequality for $p = p' = 2$ we have

$$\begin{aligned}
\int_{\partial\Omega_R} |u_{sc}| d\sigma(y) &\le \Big(\int_{\partial\Omega_R} 1 d\sigma(y)\Big)^{1/2}\Big(\int_{\partial\Omega_R} |u_{sc}(y)|^2 d\sigma(y)\Big)^{1/2} \\
&= (4\pi R^2)^{1/2}\Big(\int_{\partial\Omega_R} |u_{sc}(y)|^2 d\sigma(y)\Big)^{1/2} = O(R).
\end{aligned}$$

Since $\Delta_y G_k^+$ satisfies Sommerfeld radiation condition at the infinity, we have

$$\int_{\partial\Omega_R} u_{sc}\Big(\frac{\partial}{\partial n} - ik\Big)\Delta_y G_k^+(|x-y|)d\sigma(y) = o(1).$$

Exactly the same calculation can be done for the second term in the last integral of (7). For the rest we again use Sommerfeld radiation conditions for u_{sc} and Δu_{sc} and the fact that both G_k^+ and $\Delta_y G_k^+$ are $O(R^{-1})$ to have that the second integral in (7) is $o(1)$ as $R \to \infty$. □

3.2 Solution to the Lippmann-Schwinger equation

Let us assume that the function $V(\cdot, |u|)$ satisfies the following conditions

$$\begin{aligned} |V(\cdot,|u|)| &\leq C_\rho\, \alpha(\cdot), \quad \alpha \in L^1\ ,\ \rho \geq |u| \\ |V(\cdot, s_1) - V(\cdot, s_2)| &\leq \widetilde{C_\rho}\beta(\cdot)|s_1 - s_2|, \quad \beta \in L^1,\ \rho \geq s_1, s_2. \end{aligned} \tag{8}$$

Theorem 3.5. *Let V be a function satisfying* (8)*. Then for any $\rho > 0$ there exists $k_0 > 0$ such that the equation*

$$u(x) = e^{ik(x,\theta)} - \int_{\mathbb{R}^3} G_k^+(|x-y|)V(y,|u|)u(y)dy \tag{9}$$

has a unique solution in $B_\rho(0) = \{f \in L^\infty(\mathbb{R}^3) : \|f\|_\infty \leq \rho\}$, for all $k \geq k_0$.

Proof. In this proof we are going to use Banach fixed-point theorem. In order to do so, we need to show that for any $\rho > 0$ there exists $k_0 > 0$ such that for all $k \geq k_0$, the operator F defined by

$$Fu(x) := e^{ik(x,\theta)} - \int_{\mathbb{R}^3} G_k^+(|x-y|)V(y,|u|)u(y)dy$$

is a contraction mapping from $B_\rho(0)$ to itself. Let $\rho > 0$. Using the estimate from Lemma 3.1 and the behaviour (8), we have

$$\begin{aligned} |Fu| &= \left| e^{ik(x,\theta)} - \int_{\mathbb{R}^3} G_k^+(|x-y|)V(y,|u|)u(y)dy \right| \\ &\leq 1 + \frac{C_\rho \|u\|_\infty}{2\pi k} \int_{\mathbb{R}^3} |\alpha(y)|dy \leq 1 + \frac{C_\rho \rho}{2\pi k}\|\alpha\|_1. \end{aligned}$$

By choosing

$$k_1 \geq \frac{\rho\, C_\rho \|\alpha\|_1}{2\pi\rho - 2\pi},$$

we have that $F : B_\rho(0) \to B_\rho(0)$, for all $k \geq k_1$.

Next we show that F is a contraction. Let $u_1, u_2 \in B_\rho(0)$. Then

$$\begin{aligned} |Fu_1 - Fu_2| &= \left| \int_{\mathbb{R}^3} G_k^+(|x-y|)\big[V(y,|u_1|)u_1 - V(y,|u_2|)u_2\big]dy \right| \\ &\leq \frac{1}{2\pi k} \int_{\mathbb{R}^3} \Big| V(y,|u_1|)(u_1 - u_2) + [V(y,|u_1|) - V(y,|u_2|)]u_2 \Big| dy \\ &\leq \frac{1}{2\pi k} \int_{\mathbb{R}^3} \Big[C_\rho |\alpha(y)| + \widetilde{C_\rho}|\beta(y)|\rho \Big] |u_1 - u_2| dy \\ &\leq \frac{1}{2\pi k}\Big[C_\rho \|\alpha\|_1 + \widetilde{C_\rho}\|\beta\|_1 \rho \Big] \|u_1 - u_2\|_\infty = \tau \|u_1 - u_2\|_\infty. \end{aligned}$$

If
$$k_2 > \frac{C_\rho\|\alpha\|_1 + \widetilde{C_\rho}\|\beta\|_1\rho}{2\pi}, \quad \text{then } \tau < 1 \text{ for all } k > k_2.$$
Therefore by setting $k_0 = \max\{k_1, k_2\}$ and using Banach fixed-point theorem, we have the result. □

Remark 3.6. Banach fixed-point theorem also gives us a way to solve the equation (9). Let $u_0(x) = e^{ik(\theta,x)}$, and $u_j(x) = Fu_{j-1}$. Then
$$u(x) := \lim_{j\to\infty} u_j(x)$$
is the unique solution to (9).

Lemma 3.7. *If we assume that in addition to α satisfying (8), for some $R > 0$ the function $\alpha \in L^p(|x| < R)$, $p > \frac{3}{2}$ and $|\alpha(x)| \le C|x|^{-\mu}$, $\mu > 3$ for $|x| \ge R$ then the following norm estimate holds*
$$\|u_{sc}\|_\infty \le \frac{C}{k^2},$$
for $k > 0$ large enough.

Proof. Let us define
$$u_{sc}^j(x) = -\int_{\mathbb{R}^3} G_k^+(|x-y|)V(y, |u_{j-1}|)u_{j-1}(y)dy,$$
where u_j is as in Remark 3.6.

Now for any $j = 1, 2, \ldots$, we have
$$u_{sc}^j(x) = -\int_{\mathbb{R}^3} G_k^+(|x-y|)V(y,|u_{j-1}|)e^{ik(y,\theta)}dy$$
$$+ \int_{\mathbb{R}^3} G_k^+(|x-y|)V(y,|u_{j-1}|)\int_{\mathbb{R}^3} G_k^+(|y-z|)V(z,|u_{j-2}|)e^{ik(z,\theta)}dzdy - \ldots$$
$$+ (-1)^j \int_{\mathbb{R}^3} G_k^+(|x-x_1|)V(x_1,|u_{j-1}|)\int_{\mathbb{R}^3} G_k^+(|x_1-x_2|)V(x_2,|u_{j-2}|)\cdots$$
$$\int_{\mathbb{R}^3} G_k^+(|x_{j-1}-x_j|)V(x_j,|u_0|)e^{ik(x_j,\theta)}dx_j\cdots dx_2dx_1.$$
As a convergent sequence $\{\|u_j\|_\infty\}_{j=0}^\infty \subset \mathbb{R}$ is bounded and therefore there exists $\rho > 0$ such that $|u_j| \le \rho$ for all $j = 0, 1, \ldots$.
For any $l = 0, 1, \ldots$, the following estimate holds
$$\int_{\mathbb{R}^3} \left|G_k^+(|x-y|)V(y,|u_l|)\right|dy \le \frac{C_\rho}{4\pi k^2}\int_{\mathbb{R}^3} \frac{\alpha(y)}{|x-y|}dy$$
$$\le \frac{C_\rho}{4\pi k^2}\left[\int_{|y|\le R} \frac{\alpha(y)}{|x-y|}dy + \int_{|y|>R} \frac{C}{|y|^\mu|x-y|}dy\right]. \tag{10}$$

Now by using Hölder inequality, we have that

$$\int_{|y|\le R} \frac{|\alpha(y)|}{|x-y|}dy \le \Big(\int_{|y|\le R} |\alpha(y)|^p dy\Big)^{1/p}\Big(\int_{|y|\le R} \frac{1}{|x-y|^{p'}}dy\Big)^{1/p'}$$
$$= \|\alpha\|_p \||x-\cdot|^{-1}\|_{p'}.$$

Now because $\frac{1}{p}+\frac{1}{p'}=1$ and thus $p'<3$, the norm $\||x-\cdot|^{-1}\|_{p'}$ is finite for all $x\in\mathbb{R}^3$.
For the second integral in (10) we consider two cases. If $|x|<R/2$ then $|x-y|\ge|y|-|x|>R/2$ and

$$\int_{|y|>R} \frac{1}{|y|^\mu|x-y|}dy \le \frac{2}{R}\int_{|y|>R}\frac{1}{|y|^\mu}dy = \frac{2}{R}\int_{\mathbb{S}^2}d\theta\int_R^\infty r^{2-\mu}dr$$
$$= \frac{8\pi}{R(3-\mu)}\Big[0-R^{3-\mu}\Big] = \frac{8\pi}{(\mu-3)R^{\mu-2}}.$$

Let now $|x|\ge R/2$ and $\varepsilon>0$ such that $2<\mu-\varepsilon<3$. Then by Theorem 2.17 we have

$$\int_{|y|>R}\frac{1}{|y|^\mu|x-y|}dy \le \frac{1}{R^\varepsilon}\int_{|y|>R}\frac{1}{|y|^{\mu-\varepsilon}|x-y|}dy$$
$$\le \frac{1}{R^\varepsilon}C|x|^{3-1-(\mu-\varepsilon)} \le \frac{C}{R^{\mu-2}}.$$

Hence for any $R>0$ and $\mu>3$ there exists a constant C such that

$$\int_{\mathbb{R}^3}\Big|G_k^+(|x-y|)V(y,|u_l|)\Big|dy \le \frac{C}{k^2}$$

and therefore

$$|u^j_{sc}(x)| \le \sum_{l=1}^{j}\Big(\frac{C}{k^2}\Big)^l.$$

For $k>0$ large enough and $j\to\infty$ this sum becomes a geometric progression and

$$|u_{sc}(x)| \le \frac{C/k^2}{1-C/k^2} = \frac{1}{k^2}\frac{C}{1-C/k^2} = O(\frac{1}{k^2}),$$

uniformly in $x\in\mathbb{R}^3$. □

Lemma 3.8. *Let u^j_{sc} be the sequence defined in Lemma 3.7 and both α and β are as was α in Lemma 3.7. For all $j\in\mathbb{N}_0$ the following norm estimate holds*

$$\|u_{sc}-u^j_{sc}\|_\infty \le \frac{C}{k^2}\Big(\frac{\widetilde{C}}{k}\Big)^{2j}.$$

Proof. We will proof this Lemma by induction. First note that $u^0_{sc} = 0$ and therefore the estimate holds for $j = 0$ due to Lemma 3.7. Let us assume that

$$\|u_{sc} - u^{j-1}_{sc}\|_\infty \leq \frac{C}{k^2}\Big(\frac{\widetilde{C}}{k}\Big)^{2j-2}.$$

By using the integral expressions of functions u_{sc} and u^j_{sc} we have

$$\begin{aligned}
|u_{sc} - u^j_{sc}| = \Big| \int_{\mathbb{R}^3} & G^+_k(|x-y|)\Big[V(y, |u_0 + u_{sc}|)(u_0 + u_{sc})(y) \\
& - V(y, |u_0 + u^{j-1}_{sc}|)(u_0 + u^{j-1}_{sc})(y)\Big] dy\Big| \\
& \leq \frac{1}{4\pi k^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\Big|V(y, |u_0 + u_{sc}|) - V(y, |u_0 + u^{j-1}_{sc}|)\Big| dy \\
& + \frac{1}{4\pi k^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\Big|[V(y, |u_0 + u_{sc}|)u_{sc}(y) - V(y, |u_0 + u^{j-1}_{sc}|)u^{j-1}_{sc}(y)\Big| dy \\
& = J_1 + J_2.
\end{aligned}$$

The properties of function β and the usage of triangle inequality gives us the following estimate

$$J_1 \leq \frac{\widetilde{C}_\rho}{4\pi k^2}\int_{R^3}\frac{|\beta(y)|}{|x-y|}\Big||u_0 + u_{sc}| - |u_0 + u^{j-1}_{sc}|\Big| dy \leq \frac{C_1}{k^2}\|u_{sc} - u^{j-1}_{sc}\|_\infty.$$

For J_2 we do similar calculation as in the proof of Theorem 3.5 and we obtain

$$\begin{aligned}
J_2 = & \frac{1}{4\pi k^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\Big|V(y, |u_0 + u_{sc}|)(u_{sc} - u^{j-1}_{sc})(y) \\
& + \Big[V(y, |u_0 + u_{sc}|) - V(y, |u_0 + u^{j-1}_{sc}|)\Big]u^{j-1}_{sc}(y)\Big| dy \\
& \leq \frac{C_\rho}{4\pi k^2}\int_{\mathbb{R}^3}\frac{|\alpha(y)|}{|x-y|}|(u_{sc} - u^{j-1}_{sc})(y)| dy \\
& + \frac{\rho\widetilde{C}_\rho}{4\pi k^2}\int_{\mathbb{R}^3}\frac{|\beta(y)|}{|x-y|}\Big||u_0 + u_{sc}| - |u_0 + u^{j-1}_{sc}|\Big| dy \\
& \leq \frac{C_2}{k^2}\|u_{sc} - u^{j-1}_{sc}\|_\infty + \frac{C_3}{k^2}\|u_{sc} - u^{j-1}_{sc}\|_\infty.
\end{aligned}$$

Notice that

$$\int_{\mathbb{R}^3}\frac{|\alpha(y)|}{|x-y|}dy,\ \int_{\mathbb{R}^3}\frac{|\beta(y)|}{|x-y|}dy < \infty$$

as was shown in the proof of Lemma 3.7. By taking $\widetilde{C} = C_1 + C_2 + C_3$, the claim follows by induction. □

3.3 Asymptotics for Lippmann-Schwinger equation

Theorem 3.9. *Let* $u \in L^\infty(\mathbb{R}^3)$ *be a solution to the equation* (9)*. If in addition we assume that there exists* $R > 0$ *such that*

$$|\alpha(x)| \leq \frac{C}{|x|^\mu}, \quad \text{when } |x| > R \text{ and } \mu > 3, \tag{11}$$

then for fixed $k > 0$ *and* $|x| \to \infty$,

$$u(x,k,\theta) = e^{ik(x,\theta)} - C_3 \frac{e^{ik|x|}}{k^2|x|} A(k,\theta',\theta) + o\Big(\frac{1}{|x|}\Big), \tag{12}$$

where function $A(k,\theta',\theta)$ *is called the scattering amplitude and it is defined as*

$$A(k,\theta',\theta) = \int_{\mathbb{R}^3} e^{-ik(\theta',y)} V(y,|u|)u(y,k,\theta)dy.$$

The constant depending on the number of dimensions in our case is $C_3 = \frac{1}{8\pi}$ *and* $\theta' = \frac{x}{|x|}$ *is the direction of the observation.*

Proof. In this proof we follow the footsteps of [5] and [11]. From (6) we have a representation for the function u_{sc} as an integral over $\mathbb{R}^3$. Let us split the area of integration as

$$\begin{aligned} u_{sc}(x) &= -\int_{\mathbb{R}^3} G_k^+(|x-y|)V(y,|u|)u(y)dy \\ &= -\int_{|y|\leq|x|^\alpha} G_k^+(|x-y|)V(y,|u|)u(y)dy \\ &\quad -\int_{|y|>|x|^\alpha} G_k^+(|x-y|)V(y,|u|)u(y)dy = I_1 + I_2, \end{aligned}$$

where the value of parameter $0 < a < 1$ will be determined later. Let us first consider I_1. When $|y| \leq |x|^a$, by using the asymptotic formula $(1+x)^s = 1 + sx + O(x^2)$, we have that

$$\begin{aligned} |x-y| &= (|x|^2 - 2(x,y) + |y|^2)^{1/2} \\ &= |x|\Big[1 - \frac{(x,y)}{|x|^2} + \frac{|y|^2}{|x|^2} + O\Big(\Big[-\frac{2(x,y)}{|x|^2} + \frac{|y|^2}{|x|^2}\Big]^2\Big)\Big] \\ &= |x| - (\theta',y) + \frac{|y|^2}{2|x|^2} + O(|x|^{2a-1}) \\ &= |x| - (\theta',y) + O(|x|^{2a-1}). \end{aligned}$$

Similar calculation shows that

$$|x-y|^{-1} = |x|^{-1}[1 + O(|x|^{a-1})].$$

For fixed $k > 0$ and $|y| \le |x|^a$ we choose $a < \frac{1}{2}$. Now $2a-1 < 0$ and therefore, when $|y| \le |x|^a \le |x| \to \infty$, we can use the small argument behaviour for exponent function $e^x = 1 + O(x)$.
These three estimates gives us the following asymptotic behaviour for kernel G_k^+, when $|y| \le |x|^a$.

$$\begin{aligned}
G_k^+(|x-y|) &= \frac{1}{8\pi k^2}|x-y|^{-1}(e^{ik|x-y|} - e^{-k|x-y|}) \\
&= \frac{1}{8\pi k^2}|x|^{-1}(1 + O(|x|^{a-1}))(e^{ik|x|}e^{-ik(\theta',y)}e^{ikO(|x|^{2a-1})} \\
&- e^{-k|x|}e^{k(\theta',y)}e^{-kO(|x|^{2a-1})}) \\
&= \frac{1}{8\pi k^2|x|}(e^{ik|x|}e^{-ik(\theta',y)} - e^{-k|x|}e^{k(\theta',y)}) + O(|x|^{2a-2}). \qquad (13)
\end{aligned}$$

By substituting this into I_1, we have

$$\begin{aligned}
I_1 &= -\frac{e^{ik|x|}}{8\pi k^2|x|}\int_{\mathbb{R}^3} e^{-ik(\theta',y)}V(y,|u|)u(y)dy \\
&+ \frac{e^{-k|x|}}{8\pi k^2|x|}\int_{|y|\le|x|^a} e^{k(\theta',y)}V(y,|u|)u(y)dy \\
&+ \frac{e^{ik|x|}}{8\pi k^2|x|}\int_{|y|>|x|^a} e^{-ik(\theta',y)}V(y,|u|)u(y)dy \\
&+ O(|x|^{2a-2})\int_{|y|\le|x|^a} V(y,|u|)u(y)dy.
\end{aligned}$$

Here the first term is of desired form and the rest is $o(\frac{1}{|x|})$. Indeed in the second term we have an integral of $L^1(\mathbb{R}^3)$-function multiplied by $\frac{Ce^{-k|x|}}{|x|}$ which is clearly $o(\frac{1}{|x|})$. The integral on the third row tends to zero as $|x| \to \infty$ due to $V(\cdot,|u|)u(\cdot)$ being a $L^1(\mathbb{R}^3)$-function. Because of our choice $a < \frac{1}{2}$, the last term is also $o(\frac{1}{|x|})$.
Next we will show that $I_2 = o(\frac{1}{|x|})$. We start by splitting the integral into two parts

$$\begin{aligned}
I_2 = &-\int_{|x|^\alpha<|y|<\frac{|x|}{2}} G_k^+(|x-y|)V(y,|u|)u(y)dy \\
&-\int_{|y|>\frac{|x|}{2}} G_k^+(|x-y|)V(y,|u|)u(y)dy = I_2' + I_2''.
\end{aligned}$$

When $|x|^\alpha < |y| < |x|/2$, by triangle inequality $|x-y| \geq \big||x|-|y|\big| \geq \frac{|x|}{2}$. Therefore,

$$\begin{aligned}
|I_2'| &\leq \int_{|x|^\alpha<|y|<\frac{|x|}{2}} \Big|G_k^+(|x-y|)V(y,|u|)u(y)\Big|dy \\
&\leq C\int_{|x|^\alpha<|y|<\frac{|x|}{2}} \frac{|V(y,|u|)u(y)|}{|x-y|}dy \\
&\leq \frac{C}{|x|}\int_{|x|^\alpha<|y|<\frac{|x|}{2}} \Big|V(y,|u|)u(y)\Big|dy.
\end{aligned}$$

Because the function $V(\cdot,|u|)u(\cdot) \in L^1(\mathbb{R}^3)$, the integral above can be estimated as

$$\int_{|x|^\alpha<|y|<\frac{|x|}{2}} \Big|V(y,|u|)u(y)\Big|dy \leq \int_{|x|^\alpha<|y|} \Big|V(y,|u|)u(y)\Big|dy = o(1), \text{ as } |x| \to \infty.$$

For I_2'' we will use Theorem 2.17. Since $|\alpha(x)| \leq \frac{C}{|x|^\mu}$ and $\mu > 3$, we can take $\varepsilon > 0$ such that $2 < \mu - \epsilon < 3$.
Then

$$\begin{aligned}
|I_2''| &\leq C\int_{|y|\geq|x|/2} \frac{|V(y,|u|)|}{|x-y|}dy \leq C\int_{|y|\geq|x|/2} \frac{1}{|y|^\mu|x-y|}dy \\
&\leq \frac{C}{|x|^\varepsilon}\int_{|y|\geq|x|/2} \frac{1}{|y|^{\mu-\varepsilon}|x-y|}dy.
\end{aligned}$$

Here we have two kernels

$$K_1(x,y) = \frac{1}{|x-y|} \quad \text{and } K_2(y,z) = \frac{1}{|y|^{\mu-\varepsilon}}$$

which meet the conditions of Theorem 2.17. In our case we have $\alpha_1 = 1$ and $\alpha_2 = \mu - \varepsilon$ and due to our choice of $\varepsilon > 0$ we have $\alpha_1 + \alpha_2 = 1 + \mu - \varepsilon > 3$. Therefore by using Theorem 2.17 and Remark 2.18 we obtain

$$\begin{aligned}
|I_2''| &\leq \frac{C}{|x|^\varepsilon}\int_{|y|\geq|x|/2} \frac{1}{|y|^{\mu-\varepsilon}|x-y|}dy \leq \frac{C}{|x|^\varepsilon}\int_{\mathbb{R}^3} \frac{1}{|y|^{\mu-\epsilon}|x-y|}dy \\
&\leq \frac{C}{|x|^\varepsilon}|x|^{3-1-(\mu-\varepsilon)} = \frac{C}{|x|^{\mu-2}} = o\Big(\frac{1}{|x|}\Big).
\end{aligned}$$

Thus we have shown that both I_2' and I_2'' are $o(\frac{1}{|x|})$. This gives us that $I_2 = o(\frac{1}{|x|})$ and the theorem is proved. $\square$

3.4 Saito's formula

Theorem 3.10. *Let $\alpha, \beta \in L^p(|x| < R)$ for some $R > 0$, where $p > 3$ and when $|x| > R$ they satisfy the condition* (11)*. Then*

$$\lim_{k\to\infty} k^2 \int_{\mathbb{S}^2\times\mathbb{S}^2} e^{-ik(\theta-\theta',x)} A(k,\theta',\theta)d\theta d\theta' = 8\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2}dy, \tag{14}$$

uniformly in $x \in \mathbb{R}^3$.

Proof. We start by substituting $u = u_0 + u_{sc}$ and dividing the integral into two parts.

$$\begin{aligned} k^2 &\int_{\mathbb{S}^2\times\mathbb{S}^2} e^{-ik(\theta-\theta',x)} A(k,\theta',\theta)d\theta d\theta' \\ &= k^2 \int_{\mathbb{S}^2\times\mathbb{S}^2} e^{-ik(\theta-\theta',x)} \int_{\mathbb{R}^3} e^{-ik(\theta'-\theta,y)} V(y,|u|)dy d\theta d\theta' \\ &+ k^2 \int_{\mathbb{S}^2\times\mathbb{S}^2} e^{-ik(\theta-\theta',x)} \int_{\mathbb{R}^3} e^{-ik(\theta',y)} V(y,|u|)u_{sc}(y)dy d\theta d\theta' = I_1 + I_2. \end{aligned}$$

For I_1 we can substitute $V(y,|u|) = V(y,|u|) - V(y,1) + V(y,1)$ and further split the integration into two parts to obtain

$$\begin{aligned} I_1 = k^2 &\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} e^{-ik(\theta',y-x)}d\theta' \int_{\mathbb{S}^2} e^{-ik(\theta,x-y)} \Big[V(y,|u|) - V(y,1)\Big] d\theta dy \\ + k^2 &\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} e^{-ik(\theta',y-x)}d\theta' \int_{\mathbb{S}^2} e^{-ik(\theta,x-y)} V(y,1)d\theta dy = I_1^{(1)} + I_1^{(2)}. \end{aligned}$$

Let us first consider $I_1^{(2)}$. We would like to first integrate with respect to θ' and θ. Note that since we are integrating over the whole unit sphere, the integral does not depend on the angle $\frac{x-y}{|x-y|}$. Therefore without loss of generality we may set it to on one of the axis. When we switch to spherical coordinates, the inner product $(\theta, \frac{x-y}{|x-y|}) = \cos(\mu)$, where $\mu \in [0,\pi]$. Now the Jacobian of this transformation is $\sin(\mu)$ [4, p. 441.] and therefore we have

$$\begin{aligned} \int_{\mathbb{S}^2} e^{ik(\theta,x-y)}d\theta &= \int_0^{\pi}\int_0^{2\pi} e^{ik|x-y|\cos\mu} \sin\mu d\varphi d\mu \\ &= -\frac{2\pi}{ik|x-y|}(e^{ik|x-y|} - e^{-ik|x-y|}) = \frac{4\pi}{k|x-y|}\sin(k|x-y|). \end{aligned} \tag{15}$$

Now substituting (15) into $I_1^{(2)}$ and using a trigonometric formula we have that

$$\begin{aligned} I_1^{(2)} &= 16\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2} \sin^2(k|x-y|)dy \\ &= 8\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2} dy - 8\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2} \cos(2k|x-y|)dy. \end{aligned} \tag{16}$$

For the second term above we will use the Riemann-Lebesgue lemma to show that it tends to zero as $k \to \infty$. In order to do so we must show that

$$g(y) := \frac{V(y,1)}{|x-y|^2} \quad \text{is a } L^1(\mathbb{R}^3) - \text{function.}$$

Indeed,

$$\int_{\mathbb{R}^3} \frac{|V(y,1)|}{|x-y|^2} dy \le C \int_{|y|\le R} \frac{|\alpha(y)|}{|x-y|^2} dy + \widetilde{C} \int_{|y|>R} \frac{1}{|y|^\mu |x-y|^2} dy = K_1 + K_2.$$

For K_1 we use Hölder inequality to obtain

$$K_1 \le C\Big(\int_{|y|\le R} |\alpha(y)|^p dy\Big)^{1/p} \Big(\int_{|y|\le R} \frac{1}{|x-y|^{2p'}} dy\Big)^{1/p'} \le C\|\alpha\|_p \||x-\cdot|^{-2}\|_{p'}, \tag{17}$$

where both of these norms are finite when $p > 3$.

For K_2 we proceed as in the proof of Lemma 3.7. Let us consider two cases. When $|x| \le \frac{R}{2}$ we have $|x-y| \ge |y| - |x| \ge \frac{R}{2}$ and

$$\int_{|y|>R} \frac{1}{|y|^\mu |x-y|^2} dy \le \frac{4}{R^2} \int_{|y|>R} |y|^{-\mu} dy < +\infty. \tag{18}$$

If $|x| > \frac{R}{2}$ we can use Theorem 2.17 same way we did before in the proof of Lemma 3.7 and obtain

$$\int_{|y|>R} \frac{1}{|y|^\mu |x-y|^2} dy \le \frac{C}{|x|^{\mu-1}} < \frac{C'}{R^{\mu-1}}. \tag{19}$$

Now combining (17) - (19) we have that the function $g(y)$ is a L^1-function. Therefore by Riemann-Lebesgue lemma the last term in (16) goes to zero as $k \to \infty$ and

$$I_1^{(2)} \to 8\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2} dy, \quad \text{as } k \to \infty.$$

Next we consider $I_1^{(1)}$. Let us first split the area of integration into two parts

$$\begin{aligned}
I_1^{(1)} &= k^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-ik(\theta', y-x)} d\theta' \int_{\mathbb{S}^2} e^{-ik(\theta, x-y)} \Big[V(y, |u|) - V(y, 1) \Big] d\theta dy \\
&= 4\pi k \int_{|y| \le R} \frac{\sin(k|x-y|)}{|x-y|} \int_{\mathbb{S}^2} e^{-ik(\theta, x-y)} \Big[V(y, |u|) - V(y, 1) \Big] d\theta dy \\
&+ 4\pi k \int_{|y| > R} \frac{\sin(k|x-y|)}{|x-y|} \int_{\mathbb{S}^2} e^{-ik(\theta, x-y)} \Big[V(y, |u|) - V(y, 1) \Big] d\theta dy \\
&= J_1 + J_2.
\end{aligned}$$

Because $\beta \in L^p(|x| < R)$-function for $p > 3$ and Lemma 3.7 gives us the estimate $\|u_{sc}\|_\infty = O\Big(\frac{1}{k^2}\Big)$ by using Hölder inequality, we have

$$\begin{aligned}
|J_1| &\le 4\pi k \int_{\mathbb{S}^2} \int_{|y| \le R} \frac{|V(y, |u|) - V(y, 1)|}{|x-y|} dy d\theta \\
&\le C\, k \int_{\mathbb{S}^2} \int_{|y| \le R} \frac{|\beta(y)||u| - 1|}{|x-y|} dy d\theta \\
&\le Ck\|u_{sc}\|_\infty \int_{\mathbb{S}^2} \Big(\int_{|y| \le R} |\beta(y)|^p dy \Big)^{1/p} \Big(\int_{|y| \le R} \frac{1}{|x-y|^{p'}} dy \Big)^{1/p'} d\theta \\
&\le C\, k||u_{sc}||_\infty \|\beta\|_p \||x - \cdot|^{-1}\|_{p'} = O\Big(\frac{1}{k}\Big).
\end{aligned}$$

For J_2 we use the behaviour (11) to have

$$\begin{aligned}
|J_2| &\le 4\pi k \int_{\mathbb{S}^2} \int_{|y| > R} \frac{|V(y, |u|) - V(y, 1)|}{|x-y|} dy d\theta \\
&\le 4\pi k \int_{\mathbb{S}^2} \int_{|y| > R} \frac{|\beta(y)|\; |\; |u| - 1\; |}{|x-y|} dy d\theta \\
&\le 4\pi k||u_{sc}||_\infty \int_{\mathbb{S}^2} d\theta \int_{|y| > R} \frac{1}{|y|^\mu |x-y|} dy.
\end{aligned}$$

The last integral was shown to be finite in Lemma 3.7 for any $x \in \mathbb{R}^3$ and therefore $|J_2| = O(\frac{1}{k})$.

Finally for I_2 we can first integrate with respect to θ' by using (15) and we have

$$\begin{aligned}
I_2 &= k^2 \int_{\mathbb{S}^2 \times \mathbb{S}^2} e^{-ik(\theta - \theta', x)} \int_{\mathbb{R}^3} e^{-ik(\theta', y)} V(y, |u|) u_{sc}(y) dy d\theta d\theta' \\
&= k^2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-ik(\theta', y-x)} d\theta' \int_{\mathbb{S}^2} e^{-ik(\theta, x)} V(y, |u|) u_{sc}(y) d\theta dy \\
&= 4\pi k \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} e^{-ik(\theta, x)} \sin(k|x-y|) \frac{V(y, |u|) u_{sc}(y)}{|x-y|} dy d\theta.
\end{aligned}$$

Now this can be estimated by modulus as follows

$$
\begin{aligned}
|I_2| &= 4\pi k \Big| \int_{\mathbb{S}^2} \int_{\mathbb{R}^3} e^{-ik(\theta,x)} \sin(k|x-y|) \frac{V(y,|u|)u_{sc}(y)}{|x-y|} dy d\theta \Big| \\
&\leq C\, k\, \|u_{sc}\|_\infty \int_{\mathbb{R}^3} \frac{|\alpha(y)|}{|x-y|} dy \\
&\leq Ck\|u_{sc}\|_\infty \Big(\int_{|y|\leq R} \frac{|\alpha(y)|}{|x-y|} dy + \int_{|y|>R} \frac{\widetilde{C}}{|y|^\mu |x-y|} dy \Big).
\end{aligned} \tag{20}
$$

For the first integral above we will use Hölder inequality to obtain

$$
\begin{aligned}
\int_{|y|\leq R} \frac{|\alpha(y)|}{|x-y|} dy &\leq \Big(\int_{|y|\leq R} |\alpha(y)|^p dy \Big)^{1/p} \Big(\int_{|y|\leq R} \frac{1}{|x-y|^{p'}} dy \Big)^{1/p'} \\
&= \|\alpha\|_p \| |x-\cdot|^{-1} \|_{p'}.
\end{aligned}
$$

The second integral in (20) was shown to be finite earlier and therefore the estimate for $\|u_{sc}\|_\infty$ gives us that

$$
|I_2| = O(\frac{1}{k}).
$$

By combining what we have done, we have shown that both $I_1^{(1)}$ and I_2 go to zero as $k \to \infty$ and $I_1^{(2)}$ tends to the right hand side of (14) as $k \to \infty$.

□

4 Inverse scattering problem

In inverse scattering problem we assume that the scattering amplitude $A(k, \theta, \theta')$ is known. Our task then is to determine some characteristics of the unknown function V. Having the Saito's formula proven we obtain two corollaries, namely uniqueness and representation formula for function $V(x, 1)$. For three-dimensional Schrödinger equation the corresponding results have been proved in [5] and we will follow those proofs.

Corollary 4.1 (Uniqueness)**.** *Let $V_1(x,1)$ and $V_2(x,1)$ be as in Theorem 3.10. If the respective scattering amplitudes A_1 and A_2 coincide for some sequence $k_j \to \infty$, as $j \to \infty$, then $V_1(x,1)$ and $V_2(x,1)$ are equal in the sense of tempered distributions.*

Proof. It is enough to show that the homogeneous equation

$$
f(x) := 8\pi^2 \int_{\mathbb{R}^3} \frac{V(y,1)}{|x-y|^2} dy = 0 \tag{21}
$$

has only a trivial solution $V(\cdot,1)\equiv 0$ in the sense of tempered distributions. Note that the function f is a convolution of functions $V(x,1)$ and $\frac{1}{|x|^2}$. Therefore we can calculate its Fourier transform as

$$\begin{aligned}\mathcal{F}(f)(\xi) &= 8\pi^2\mathcal{F}(V(x,1)*\frac{1}{|x|^2})(\xi)\\ &= 8\pi^2(2\pi)^{3/2}\mathcal{F}(V(x,1))(\xi)\mathcal{F}\left(\frac{1}{|x|^2}\right)(\xi).\end{aligned} \tag{22}$$

The value of $\mathcal{F}(|x|^{-2})$ can be calculated precisely by using (15) and we have

$$\begin{aligned}\mathcal{F}\Big(\frac{1}{|x|^2}\Big)(\xi) &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i(x,\xi)}\frac{1}{|x|^2}dx = (2\pi)^{-3/2}\int_0^\infty\int_{\mathbb{S}^2} e^{-ir(\theta',\xi)}d\theta' dr\\ &= (2\pi)^{-3/2}\int_0^\infty \frac{4\pi}{r|\xi|}\sin(r|\xi|)dr = \frac{4\pi}{(2\pi)^{3/2}|\xi|}\int_0^\infty \frac{\sin(t)}{t}dt\\ &= \frac{2\pi^2}{(2\pi)^{3/2}|\xi|}.\end{aligned} \tag{23}$$

By combining (22) and (23) we have that

$$\mathcal{F}(f)(\xi) = 16\pi^4\frac{1}{|\xi|}\mathcal{F}(V(x,1))(\xi). \tag{24}$$

Let us denote the space of $S(\mathbb{R}^3)$- functions vanishing on some neighbourhood of origin by $S_0(\mathbb{R}^3)$. Now the function f defines a tempered distribution in $S_0(\mathbb{R}^3)$.

Using (21) and (24) we have that for the Fourier transform of f the following holds

$$\begin{aligned}0 = \langle \mathcal{F}f,\psi\rangle &= 16\pi^4\langle |\xi|^{-1}\mathcal{F}(V(x,1))(\xi),\psi(\xi)\rangle\\ &= 16\pi^4\langle \mathcal{F}(V(x,1))(\xi),|\xi|^{-1}\psi(\xi)\rangle,\end{aligned}$$

for all $\psi\in S_0(\mathbb{R}^3)$.

If $\mu\in S_0(\mathbb{R}^3)$, then also $|\xi|\mu\in S_0(\mathbb{R}^3)$. Therefore

$$\langle \mathcal{F}(V(x,1))(\xi),\mu(\xi)\rangle = \langle \mathcal{F}(V(x,1))(\xi),|\xi|^{-1}|\xi|\mu(\xi)\rangle = 0,$$

for all $\mu\in S_0(\mathbb{R}^3)$. This implies that the support of $\mathcal{F}(V(x,1))$ is at most the origin. In the case where $\operatorname{supp}\mathcal{F}(V(x,1))=\{0\}$, by Theorem 2.13 there exists an integer $N\in\mathbb{N}$ and complex numbers C_α such that

$$\mathcal{F}(V(x,1)) = \sum_{|\alpha|\le N} C_\alpha\partial^\alpha\delta_0.$$

Now the inverse Fourier transform gives us that the function $V(x,1)$ is a polynomial. However no polynomial satisfies to the conditions (11) other than $V(x,1)\equiv 0$. $\square$

Corollary 4.2 (Representation formula)**.** *If the conditions from Theorem 3.10 are satisfied, then*

$$V(x,1)=\frac{1}{16\pi^4}\lim_{k\to\infty}k^3\int_{\mathbb{S}^2\times\mathbb{S}^2}A(k,\theta,\theta')|\theta-\theta'|e^{-ik(\theta-\theta',x)}d\theta d\theta',$$

in the sense of tempered distributions.

Proof. Let the function f be as in the proof of Corollary 4.1. From (24) by using the inverse Fourier transform we have

$$V(x,1)=\frac{1}{16\pi^4}\mathcal{F}^{-1}\Big(|\xi|\mathcal{F}(f)(\xi)\Big)(x). \tag{25}$$

Next we are going to calculate the Fourier transform of the function f in the sense of distributions using Saito's formula. Let $\psi\in S(\mathbb{R}^3)$. Fubini's theorem allows us to change the order of integration and we obtain

$$\begin{aligned}\langle\mathcal{F}(f),\psi\rangle=\langle f,\mathcal{F}(\psi)\rangle&=\int_{\mathbb{R}^3}\Big(\lim_{k\to\infty}k^2\int_{\mathbb{S}^2\times\mathbb{S}^2}e^{-ik(\theta-\theta',x)}A(k,\theta',\theta)d\theta d\theta'\Big)\\ &\times\int_{\mathbb{R}^3}e^{-i(\xi,x)}\psi(\xi)d\xi dx\\ &=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\lim_{k\to\infty}k^2\int_{\mathbb{S}^2\times\mathbb{S}^2}e^{-i(x,\xi+k(\theta-\theta'))}A(k,\theta',\theta)d\theta d\theta' dx\,\psi(\xi)d\xi.\end{aligned}$$

Inside the integral the only term depending on $x\in\mathbb{R}^3$ is the exponential function and therefore

$$\int_{\mathbb{R}^3}e^{-i(x,\xi+k(\theta-\theta'))}dx=\mathcal{F}(1)(\xi+k(\theta-\theta'))=(2\pi)^{3/2}\delta_0(\xi+k(\theta-\theta')),$$

where δ_0 is the Dirac delta distribution. Hence we may conclude that

$$\mathcal{F}(f)(\xi)=\lim_{k\to\infty}k^2\int_{\mathbb{S}^2\times\mathbb{S}^2}(2\pi)^{3/2}\delta_0(\xi+k(\theta-\theta'))A(k,\theta',\theta)d\theta d\theta', \tag{26}$$

in the sense of tempered distributions. Now by (25) and (26) we have that

$$\begin{aligned}\langle V(x,1),\psi(x)\rangle&=\frac{1}{16\pi^4}\Big\langle\mathcal{F}^{-1}\Big(|\xi|\mathcal{F}(f)(\xi)\Big)(x),\psi(x)\Big\rangle\\ &=\frac{1}{16\pi^4}\Big\langle|\xi|\mathcal{F}(f)(\xi),\mathcal{F}^{-1}(\psi)(\xi)\Big\rangle\\ &=\frac{1}{16\pi^4}\int_{\mathbb{R}^3}|\xi|\mathcal{F}(f)(\xi)\mathcal{F}^{-1}(\psi)(\xi)d\xi\\ &=\frac{1}{16\pi^4}\int_{\mathbb{R}^3}\lim_{k\to\infty}k^2\int_{\mathbb{S}^2\times\mathbb{S}^2}|\xi|\delta_0(\xi+k(\theta-\theta'))A(k,\theta',\theta)d\theta d\theta'\\ &\times\int_{\mathbb{R}^3}e^{i(\xi,x)}\psi(x)dxd\xi.\end{aligned}$$

Note that in the last equality the constant $(2\pi)^{3/2}$ from (26) and the constant from inverse Fourier transform cancel each other out.
By using Fubini's theorem and the fact that

$$\int_{\mathbb{R}^3} |\xi|\delta_0(\xi + k(\theta - \theta'))e^{i(\xi,x)}d\xi = k|\theta - \theta'|e^{-ik(x,\theta-\theta')},$$

we can write

$$\langle V(x,1),\psi(x)\rangle = \frac{1}{16\pi^4}\int_{\mathbb{R}^3} \lim_{k\to\infty} k^3 \int_{\mathbb{S}^2\times\mathbb{S}^2} e^{-ik(x,\theta-\theta')}|\theta - \theta'|A(k,\theta,\theta')d\theta d\theta'\psi(x)dx,$$

or in other words,

$$V(x,1) = \frac{1}{16\pi^4}\lim_{k\to\infty} k^3 \int_{\mathbb{S}^2\times\mathbb{S}^2} e^{-ik(x,\theta-\theta')}|\theta - \theta'|A(k,\theta,\theta')d\theta d\theta'$$

in the sense of tempered distributions. □

4.1 Inverse Born approximation

In this section we take a short look at one method for solving the inverse problem. The method we choose to use is called Born approximation. The idea is to define a function that approximates our unknown function $V(x,1)$ and show that the difference between the two is in some sense smoother than the function $V(x,1)$. We will give the definition for the inverse Born approximation with full scattering data and in section 4.2 we discuss the case when only the backscattering data is available. We proceed as in [6, p.493-495] and [7]. For technical reasons, for negative values of k we define

$$u(x,-k,\theta) = \overline{u(x,k,\theta)},\ k < 0.$$

Using this definition we are able to extend A to the whole line $k \in \mathbb{R}$, via

$$A(k,\theta',\theta) = \overline{A(-k,\theta',\theta)}, \quad \text{when } k < 0.$$

The estimate from Lemma 3.7 suggests us to approximate the solution to the Lippmann-Schwinger equation by the incident wave, $u \approx u_0$. When we substitute this into to the definition of the scattering amplitude, we have

$$A(k,\theta',\theta) \approx \int_{\mathbb{R}^3} e^{-ik(\theta'-\theta,y)}V(y,1)dy = (2\pi)^{3/2}\mathcal{F}\Big(V(x,1)\Big)(k(\theta'-\theta)).$$

This implies that

$$V(x,1) \approx (2\pi)^{-3/2}\mathcal{F}^{-1}\left(A(k,\theta',\theta)\right)(x), \tag{27}$$

where the inverse Fourier transform is understood in some special sense. Let us define two cylinders $M_0 = \mathbb{R} \times \mathbb{S}^2$ and $M = M_0 \times \mathbb{S}^2$ and the measures μ_θ and μ on M_0 and M, respectively, as

$$d\mu_\theta(k, \theta') = \frac{1}{4}|k|^2 dk |\theta - \theta'|^2 d\theta'$$

and

$$d\mu(k, \theta', \theta) = \frac{1}{4\pi} d\theta d\mu_\theta(k, \theta').$$

Here $d\theta$ and $d\theta'$ are regular Lebesgue measures on the unit sphere $\mathbb{S}^2$. We define the inverse Fourier transforms on these cylinders as

$$\mathcal{F}^{-1}_{M_0}(f)(x) = \frac{1}{(2\pi)^{3/2}} \int_{M_0} e^{-ik(\theta-\theta', x)} f(k, \theta') d\mu_\theta,$$
$$\mathcal{F}^{-1}_{M}(g)(x) = \frac{1}{(2\pi)^{3/2}} \int_{M} e^{-ik(\theta-\theta', x)} g(k, \theta', \theta) d\mu.$$

Now recall that the regular inverse Fourier transform is defined as

$$\mathcal{F}^{-1} f(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i(\xi, x)} f(\xi) d\xi.$$

If we fix the value of θ and write $\xi = k(\theta - \theta')$, then k and θ' can be obtained as

$$k = \frac{|\xi|}{2(\theta, \hat{\xi})}, \quad \theta' = \theta - 2(\theta, \hat{\xi})\hat{\xi}, \quad \hat{\xi} = \frac{\xi}{|\xi|}.$$

Since the Jacobian of this transformation is $\frac{1}{4}|k|^2 dk|\theta - \theta'|^2 d\theta'$, it can be shown that for the coordinate mapping

$$u_\theta : M_0 \to \mathbb{R}^3, \; u_\theta(k, \theta') = k(\theta - \theta'),$$

$$\mathcal{F}^{-1}_{M_0}(f \circ u_\theta) = \mathcal{F}^{-1} f,$$

when $f \in S$ is even and

$$\mathcal{F}^{-1}_{M}(f \circ u_\theta) = \mathcal{F}^{-1} f,$$

when $f \in S$. Here we have used the notation $(f \circ u_\theta)(\cdot)$ to denote the composition $f(u_\theta(\cdot))$. Now the approximation (27) suggests us to introduce the following definitions.

Definition 4.3. The inverse Born approximations V_B^θ and V_B are defined as

$$V_B^\theta(x) = (2\pi)^{-3/2}\mathcal{F}_{M_0}^{-1}\left(A(k,\theta',\theta)\right)(x) = \frac{1}{8\pi^3}\int_{M_0} e^{-ik(\theta-\theta',x)}A(k,\theta',\theta)d\mu_\theta$$

and

$$V_B(x) = (2\pi)^{-3/2}\mathcal{F}_{M}^{-1}\left(A(k,\theta',\theta)\right)(x) = \frac{1}{8\pi^3}\int_{M} e^{-ik(\theta-\theta',x)}A(k,\theta',\theta)d\mu,$$

in the sense of distributions.

Previous studies have shown that the function V_B can be used to determine some characteristics of the unknown function $V(x,1)$ and we expect that to be the case here as well. This definition assumes that the whole scattering data is known. However, since in practise having the whole scattering data is rarely the case, next we consider what kind of results can be obtained, when the measurements are obtained from a fixed direction, namely from the direction of the incident wave.

4.2 Inverse backscattering problem

Let us now consider the inverse backscattering problem. We assume that the angle of observation is the opposite to that of the incident wave, i.e., $\theta' = -\theta$. By approximating the function u again as $u \approx u_0$ and substituting these into the definition of scattering amplitude we obtain

$$A_b(k,\theta) = A(k,-\theta,\theta) \approx \int_{\mathbb{R}^3} e^{2ik(\theta,y)}V(y,1)dy = (2\pi)^{3/2}\mathcal{F}^{-1}\left(V(x,1)\right)(2k\theta).$$

This expression suggests us to define the inverse backscattering Born approximation as

$$V_B^b(x) := \mathcal{F}\Big((2\pi)^{-3/2}A_b(\frac{k}{2},\theta)\Big)(x) = \frac{1}{8\pi^3}\int_0^\infty k^2\int_{\mathbb{S}^2} e^{-ik(\theta,x)}A_b\Big(\frac{k}{2},\theta\Big)d\theta dk.$$

In order to simplify some future calculations we set $A_b(k,\theta) = 0$ when $k < k_0$, where $k_0 > 0$ is large enough. Therefore the Born approximation becomes

$$V_B^b(x) = \frac{1}{8\pi^3}\int_{2k_0}^\infty k^2\int_{\mathbb{S}^2} e^{-ik(\theta,x)}A_b\Big(\frac{k}{2},\theta\Big)d\theta dk.$$

Next recall that in Lemma 3.7 we obtained the solution to the integral equation as the limit

$$u(x,k,\theta) = e^{ik(\theta,x)} + \lim_{j\to\infty} u_{sc}^j(x,k,\theta).$$

Using this, we define a new sequence

$$A_b^j(k,\theta) = \int_{\mathbb{R}^3} e^{ik(\theta,y)} V(y, |u_0 + u_{sc}^j|)(u_0 + u_{sc}^j)(y,k,\theta)dy,$$

when $k \geq k_0$ and $A_b^j(k,\theta) = 0$ otherwise.

Lemma 4.4. *If $\alpha, \beta \in L^p(|x| \leq R) \cap L^1(\mathbb{R}^3)$ for some $R > 0$ and $p > 3$ and when $|x| > R$ they satisfy the condition (11), then*

$$|A_b(k,\theta) - A_b^j(k,\theta)| \leq C \frac{(\widetilde{C})^j}{k^{2j+2}}$$

Proof. Let us consider the nontrivial case, where $k > k_0$. The modulus of the difference can be estimated as

$$\begin{aligned}
&|A_b(k,\theta) - A_b^j(k,\theta)| \\
&\leq \int_{\mathbb{R}^3} \Big| V(y, |u_0 + u_{sc}|)(u_0 + u_{sc})(y,k,\theta) - V(y, |u_0 + u_{sc}^j|)(u_0 + u_{sc}^j)(y,k,\theta) \Big| dy \\
&\leq \int_{\mathbb{R}^3} \Big| V(y, |u_0 + u_{sc}|) - V(y, |u_0 + u_{sc}^j|) \Big| dy \\
&+ \int_{\mathbb{R}^3} \Big| V(y, |u_0 + u_{sc}|) u_{sc}(y,k,\theta) - V(y, |u_0 + u_{sc}^j|) u_{sc}^j(y,k,\theta) \Big| dy \\
&= K_1 + K_2.
\end{aligned}$$

Now using the Lipschitz property of function V we can estimate K_1 as

$$K_1 \leq \int_{\mathbb{R}^3} \widetilde{C}_\rho |\beta(y)| \Big| |u_0 + u_{sc}| - |u_0 + u_{sc}^j| \Big| dy \leq \widetilde{C}_\rho \|\beta\|_1 \|u_{sc} - u_{sc}^j\|_\infty.$$

For K_2 we use the Lipschitz property again and the fact that both α and β are L^1 functions to obtain

$$\begin{aligned}
K_2 &= \int_{\mathbb{R}^3} \Big| V(y, |u_0 + u_{sc}|)(u_{sc} - u_{sc}^j)(y,k,\theta) \\
&\quad + \Big[V(y, |u_0 + u_{sc}|) - V(y, |u_0 + u_{sc}^j|) \Big] u_{sc}^j(y,k,\theta) \Big| dy \\
&\leq \int_{\mathbb{R}^3} C_\rho |\alpha(y)| |u_{sc} - u_{sc}^j| dy + \int_{\mathbb{R}^3} \rho \widetilde{C}_\rho |\beta(y)| \Big| |u_0 + u_{sc}| - |u_0 + u_{sc}^j| \Big| dy \\
&\leq \Big[C_\rho \|\alpha\|_1 + \rho \widetilde{C}_\rho \|\beta\|_1 \Big] \|u_{sc} - u_{sc}^j\|_\infty.
\end{aligned}$$

Combining these estimates with the estimate from Lemma 3.8 gives us the result. □

The inverse Born sequence is given by

$$V_{B,j}^b(x) = \mathcal{F}\Big((2\pi)^{-3/2} A_b^j(\frac{k}{2},\theta)\Big)(x).$$

The study continues by showing that there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, the difference $V_B^b(x) - V_{B,j}^b(x)$ defines in some sense smooth function g. This smoothness can mean for example that the function g is continuous or it belongs to some Sobolev space. If it can be shown that such term $V_{B,j}^b(x)$ can be expressed as sum of the unknown function $V(x,1)$ and some well-behaved function f, this then implies that

$$V_B^b(x) = V(x,1) + h(x), \tag{28}$$

where h is well-behaved. Assuming that the function h is continuous we would be able to conclude that the singularities and jump discontinuities of the unknown function $V(x,1)$ would coincide with those of the inverse Born approximation $V_B^b(x)$.

For an operator with zero and first order quasi-linear perturbations on the line, it was shown in [12] that with some additional regularity conditions for perturbations the function corresponding to h was continuous and vanished at the infinity. Although in [12], on the right hand side of (28) there was a transformed version of the unknown function, it was concluded that singularities of that unknown function can be recovered from the backscattering data. In [9, 10], the inverse Born approximation was used to show that in two- and three-dimensional cases, the backscattering data is sufficient for recovering the singularities of a combination of zero and first order linear perturbations. There the approach was slightly different due to the linearity of the coefficients, but the main idea was similar to ours.

Conclusion

Direct and inverse scattering problems for three-dimensional biharmonic operator were studied. We managed to prove the existence of the solution for the direct scattering problem using integral equations. With some additional conditions for potential, we provided two norm estimates for this solution. The behaviour of the solution was shown to satisfy certain asymptotic presentation which led us to define the scattering amplitude. Saito's formula for our operator was proved and it gave us two corollaries regarding the inverse scattering problem. We gave precise proofs for these two corollaries and concluded the text by introducing the inverse backscattering Born approximation and we discussing some results that have been obtained using that approach.

In future works we will study the inverse backscattering Born approximation for this operator more carefully. We expect to be able to prove similar results as in [12, 10] for our operator. We are also interested in studying the same problem in n-dimensional space and adding a first order perturbation into the operator.

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